(33) If $y = (\sin^{-1}x)^2$ prove that

(i) $(1-x^2)y_2 - xy_1 - 2 = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

Ans. → $\because y = (\sin^{-1}x)^2$

D.b.S. w.r.t. x, we have

$y_1 = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$

or, $y_1\sqrt{1-x^2} = 2\sin^{-1}x$

Squaring both sides, we have

$y_1^2(1-x^2) = 4(\sin^{-1}x)^2$

$y_1^2(1-x^2) = 4.y$

Again D.b.S. w.r.t. x, we have

$2y_1\cdot y_2(1-x^2) + y_1^2 \times -2x = 4y_1$

$\cancel{2}y_1[y_2(1-x^2) - xy_1] = \cancel{4}^{2}\cancel{y_1}$

or, $y_2(1-x^2) - xy_1 - 2 = 0$

Diff. n times by Leibnitz's theorem, we have

$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1} \times -2x + {}^nC_2\, y_n \times -2 - [y_{n+1}x + {}^nC_1\, y_n] = 0$

or, $y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{n(n-1)}{\cancel{2}} \times y_n \times \cancel{2} - y_{n+1}x - ny_n = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - [n^2 - \cancel{n} + \cancel{n}]y_n = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - n^2y_n = 0$ proved.

36. Q.N. → If $y = \cosh(\sin^{-1}x)$, prove that

(i) $(1-x^2)y_2 - xy_1 - y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)y_n$.

Ans. → $\because y = \cosh(\sin^{-1}x)$

D. b. S. w. r. t. x, we have,

$$y_1 = \sinh(\sin^{-1}x) \times \frac{1}{\sqrt{1-x^2}}$$

or, $y_1\sqrt{1-x^2} = \sinh(\sin^{-1}x)$.

Squaring both sides, we have,

$$y_1^2(1-x^2) = \sinh^2(\sin^{-1}x)$$

$$y_1^2(1-x^2) = \left[\cosh^2(\sin^{-1}x) - 1\right]$$

$$y_1^2(1-x^2) = y^2 - 1$$

Again D. b. S. w. r. t. x, we have,

$$2y_1y_2(1-x^2) + y_1^2 \times -2x = 2yy_1$$

$$y_2(1-x^2) - xy_1 = y$$

or, $\cancel{y_2}(1-x^2)y_2 - xy_1 - y = 0$

Diff. n times by Leibnitz's theorem, we hav

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1} \times -2x + {}^nC_2 y_n \times -2 - \left[y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1\right] - y$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2} \times y_n \times 2 - y_{n+1} \cdot x + ny_n -$

or, $(1-x^2)y_{n+2} - xy_{n+1}(2n+1) - [n^2 - \cancel{n} + \cancel{n+1}]y_n = 0$

~~or, $(1-x^2)y_{n+2} - xy_{n+1}(2n+1) - (n^2+1)y_n$ Proved.~~

Hence, $(1-x^2)y_{n+2} - xy_{n+1}(2n+1) = (n^2+1)y_n$

Proved.

(38) (i) If $y = \sin mx + \cos mx$, prove that $y_n = m^n[1+(-1)^n \sin 2mx]^{1/2}$.

Ans. → $\because y = \sin mx + \cos mx$

$\therefore y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$

$y_n = m^n\left[\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right]$

$= m^n\left[\left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}^2\right]^{1/2}$

$= m^n\left[1 + 2\sin\left(mx + \frac{n\pi}{2}\right)\cdot\cos\left(mx + \frac{n\pi}{2}\right)\right]^{1/2}$

$= m^n\left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right)\right]^{1/2}$

$= m^n\left[1 + \sin(2mx + n\pi)\right]^{1/2}$

$= m^n\left[1 \pm \sin 2mx\right]^{1/2}$

$= m^n\left[1 + (-1)^n \sin 2mx\right]^{1/2}$

when n is even or n is odd

Proved.

(45) Q No.→ If $y = \frac{1}{x^2+1}$ then find y_n

Ans.→ $y = \frac{1}{x^2+1} = \frac{1}{x^2 - i^2}$

$= \frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$

$\therefore y = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$

$y_n = \frac{1}{2i}\left[\frac{(-1)^n \lfloor n}{(x-i)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+i)^{n+1}}\right]$

$= \frac{(-1)^n . \lfloor n}{2i}\left[\frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}}\right]$

put $x = r\cos\theta$ — ①

$1 = r\sin\theta$ — ②

Squaring and adding ① and ②, we have

$x^2 + 1^2 = r^2$

$r = (x^2+1^2)^{1/2}$

② ÷ ①

$\tan\theta = \frac{1}{x}$

$\theta = \tan^{-1}\frac{1}{x}$

$$y_n = \frac{(-1)^n \lfloor n}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n \lfloor n}{2i}\left[\frac{1}{r^{n+1}[\cos\theta - i\sin\theta]^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{2i\cdot r^{n+1}}\left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{2ia(x^2+1^2)^{\frac{n+1}{2}}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta - i\sin(n+1)\theta\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{2ia(x^2+1^2)^{\frac{n+1}{2}}}\left[\cancel{\cos(n+1)\theta} + i\sin(n+1)\theta - \cancel{\cos(n+1)\theta} + i\sin(n+1)\theta\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{\cancel{2i}a(x^2+1^2)^{\frac{n+1}{2}}} \times \cancel{2i}\sin(n+1)\theta$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{a(x^2+1^2)^{\frac{n+1}{2}}} \cdot \sin(n+1)\theta$$ **Ans.**

(46) QN. → If $y = \frac{1}{x^2+a^2}$, find y_n

Ans. → $y = \frac{1}{x^2+a^2} = \frac{1}{x^2 - i^2a^2}$

$$= \frac{1}{(x+ia)(x-ia)}$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

D. b. S. ~~w.r.t~~ n times with respect to x, we have

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n . \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n . \lfloor n}{(x+ia)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Put $x = r\cos\theta$ — (1)

$a = r\sin\theta$ — (2)

Squaring and adding (1) and (2), we have

$x^2 + a^2 = r^2$

$\therefore r = (x^2+a^2)^{1/2}$

(2) ÷ (1)

$\tan\theta = \frac{a}{x}$

$\theta = \tan^{-1}\frac{a}{x}$

$$y_n = \frac{(-1)^n . \lfloor n}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n . \lfloor n}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n . \lfloor n}{2ia . r^{n+1}}\left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}\right]$$

$$y_n = \frac{(-1)^n . \lfloor n}{2ia(x^2+a^2)^{\frac{n+1}{2}}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta - i\sin(n+1)\theta\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{2ia(x^2+a^2)^{\frac{n+1}{2}}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cancel{\cos(n+1)\theta} + i\sin(n+1)\theta\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{\cancel{2i}a(x^2+a^2)^{\frac{n+1}{2}}} \times \cancel{2i}\sin(n+1)\theta$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{a(x^2+a^2)^{\frac{n+1}{2}}} \cdot \sin(n+1)\theta.$$

QN.→ If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{a}\right)^n$, prove that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + 2n^2 y_n = 0$$

Ans.→ $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{a}\right)^n$

or, $\cos^{-1}\frac{y}{b} = n\log\frac{x}{a} = n\left[\log x - \log a\right]$

D.b.S.w.r.t. x, we have,

$$-\frac{1}{\sqrt{1-\frac{y^2}{b^2}}} \times \frac{1}{b} y_1 = n\left[\frac{1}{x} - 0\right]$$

or, $-\dfrac{1}{\sqrt{\frac{b^2-y^2}{b^2}}} \times \dfrac{1}{b} y_1 = \dfrac{n}{x}$

or, $-\dfrac{1}{\frac{\sqrt{b^2-y^2}}{b}} \times \dfrac{1}{b} y_1 = \dfrac{n}{x}$

or, $-\dfrac{1}{1} \times \dfrac{\cancel{b}}{\sqrt{b^2-y^2}} \times \dfrac{1}{\cancel{b}} y_1 = \dfrac{n}{x}$

or, $-\dfrac{y_1}{\sqrt{b^2-y^2}} = \dfrac{n}{x}$

or, $-xy_1 = n\sqrt{b^2-y^2}$

On squaring, we have

$x^2y_1^2 = n^2(b^2-y^2)$

Again d.b.s. w.r.t. x, we have

$x^2 \times 2y_1 \cdot y_2 + y_1^2 \times 2x = n^2(0-2y \cdot y_1)$

or, $x^2 2y_1y_2 + 2xy_1^2 = -n^2 2y \cdot y_1$

Dividing by $2y_1$, we have

$x^2y_2 + xy_1 = -n^2y$

or, $x^2y_2 + xy_1 + n^2y = 0$

Diff. n times by Leibnitz theorem, we have

$y_{n+2} \cdot x^2 + {}^nC_1 y_{n+1} \cdot 2x + {}^nC_2 y_n \cdot 2 + y_{n+1} \cdot x + {}^nC_1 y_n \cdot 1 + n^2y_n = 0$

or, $y_{n+2} \cdot x^2 + 2nxy_{n+1} + \dfrac{n(n-1)}{\cancel{2}} \cdot y_n \cdot \cancel{2} + xy_{n+1} + ny_n + n^2y_n = 0$

~~or, $y_{n+2}x^2$~~

or, $x^2y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2 - \cancel{n} + \cancel{n} + n^2) = 0$

or, $x^2y_{n+2} + xy_{n+1}(2n+1) + y_n 2n^2 = 0$

or, $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$